1.1. Differential forms.

a) Show that a closed n-form ω on the unit sphere $S^n \subset \mathbb{R}^{n+1}$ is exact if and only if $\int_{S^n} \omega = 0$.

b) Prove that a compactly supported n-form ω on $\mathbb{R}^n$ is the exterior derivative of a compactly supported $(n-1)$-form on $\mathbb{R}^n$ if and only if $\int_{\mathbb{R}^n} \omega = 0$.
Hint: Use stereographic projection to carry ω to a form ω' on S^n. By part a), $\omega' = d\nu$. Now $d\nu$ is 0 in a contractible neighborhood U of the north pole N. Use this to find an $n-2$ form μ on S^n such that $\nu = d\mu$ near N. Then $\nu - d\mu$ is zero near N, so it pulls back to a compactly supported form on $\mathbb{R}^n$.

1.2. De Rham cohomology of T^2. Determine the De Rham cohomology of the torus T^2.

1.3. Tensor fields. Let T be a $(1,2)$-tensor field on M^m. Let (φ, U) and (ψ, U) be two charts on M. Show that the component ${}^{\psi}T^c_{ab}$ of T with respect to ψ depends on the components ${}^{\varphi}T^k_{ij}$ of T with respect to φ by the following relation:

$$ {}^{\psi}T^c_{ab} = \sum_{i,j,k=1}^{m} \frac{\partial \psi^c}{\partial \varphi^k} \frac{\partial \varphi^i}{\partial \psi^a} \frac{\partial \varphi^j}{\partial \psi^b} \, {}^{\varphi}T^k_{ij}. $$